THE CLASSICAL LIMIT OF BOHMIAN MECHANICS THROUGH COHERENT STATES

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ABSTRACT. Coherent states have been studied in Quantum Mechanics as a candidate for states that exhibit classical-like behavior because such states minimize the uncertainty relation making $\Delta x \Delta p = \frac{\hbar}{2}$. The purpose of this paper is to examine the Bohmian trajectories of particles with wavepackets in coherent states. Through this analysis we see that the trajectories particles with wavefunctions in the coherent state give the classical result for the free particle, a particle under constant force, and the harmonic oscillator.

1. INTRODUCTION

Bohmian Mechanics is a mechanics for point particles with mass. It uses the basic formulas of Quantum Mechanics along with an extra equation known as the "Guidance Equation". The Guidance Equation requires the use of of particle positions and gives a complete description of each particle's trajectory. Thus, in Bohmian Mechanics, if given the particle position and its corresponding wavefunction ψ, then using the equations of Bohmian Mechanics one can compute the trajectory of the particle for all time. Furthermore, the motions of an ensemble of particles in Bohmian Mechanics lead to the statistical results one would expect when using Quantum Mechanics. For these reasons, Bohmian Mechanics has become known as the "causal interpretation of Quantum Mechanics".

Due to the prominence of Quantum Mechanics and the difficulty of Bohmian Mechanics, most publications examine Bohmian Mechanics through a high-level view: looking in detail at proofs of the compatibility of Bohmian Mechanics with the experimental results that have been used to confirm Quantum Mechanics or proving the viability of Bohmian Mechanics through the derivation of Bohmian Mechanics from accepted first-order principles. Another topic of interest has been the computation of trajectories for ensembles of particles in systems of interest from Quantum Mechanics, for example the infinite square well. However, this paper is not concerned with these topics. Instead it will examine the closed form solutions for Bohmian particles with Gaussian wavepackets in systems of interest from Quantum Mechanics.

2. FOUNDATIONS OF BOHMIAN MECHANICS

Bohmian Mechanics is a hidden-variable theory with well-defined particle trajectories that is compatible with non-relativistic Quantum Mechanics. A Bohmian system is defined by the positions of its N particles $Q_1,...Q_N$, $Q_t \in \mathbb{R}^3$ together with the mass of the kth particle m_k, and the system's corresponding wavefunction ψ, which is defined as a function

$$\psi : \mathbb{R}^{3N} \times \mathbb{R} \to \mathbb{C}$$

which governs the motion of the particles on configuration space $\mathbb{R}^{3N}$ through the formula

$$\frac{dQ_k}{dt} = v_k = \frac{\hbar}{m_k}\Im\{\frac{\nabla\psi_k}{\psi_k}(Q,t)\}$$

where $\Im\{\frac{\nabla\psi_k}{\psi_k}(Q,t)\}$ denotes the imaginary part of $\frac{\nabla\psi_k}{\psi_k}(Q,t)$ [3]. The last requirement is that the wavefunction obeys the Schrodinger equation, that is

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}\hat{H}\psi$$

where $\hat{H}$ is the non-relativistic Hamiltonian.

The last important aspect of Bohmian Mechanics is known as the Quantum Equilibrium Hypothesis. The Quantum Equilibrium Hypothesis states that for an ensemble of identical systems each having a wavefunction ψ, the typical empirical distribution of the configurations of the particles is given approximately by the probabilities $p = |\psi|^2$ (this can also be taken as a theorem, derivations from Bohmian Mechanics [4, 3]). Thus the statistical outcomes for large ensembles of particles in identical systems for Bohmian Mechanics is in accord with Born's Law from Quantum Mechanics and is thus experimentally equivalent.

It is important to notice that Bohmian Mechanics is basically an extension of Quantum Mechanics where one takes the claim that particle positions exist (though are unknown) and have well-defined trajectories.

3. The Foundational Equations in One Dimension

For the purposes of our investigation, we will look at a single particle in one dimension. For such systems,

$$v(x,t) = \frac{\hbar}{m}\Im\{\frac{\frac{\partial\psi(x,t)}{\partial x}}{\psi(x,t)}\}$$

where ψ is subject to the Schrodinger Equation

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}\hat{H}\psi.$$

With just these two equations we can solve the trajectories of single particles in one dimension with their associated wavefunction.

4. Coherent States and The Gaussian Wavepacket

Glauber Coherent States, or Coherent States, have been a subject of study since Schrodinger first identified them as being the states which minimize the uncertainty relation $\Delta x\Delta p =\geq \frac{\hbar}{2}$[1, 2]. This leads to the speculation that coherent states in Bohmian Mechanics may have classical trajectories.

Our analysis will utilize what is known as the Gaussian wavepacket. The Gaussian wavepacket is defined as a wavepacket with an initial distribution as a Gaussian, that is

$$\psi(x,0) = \frac{A}{\sqrt{\sigma}}e^{\frac{-x^2}{2\sigma^2}}$$

where $A = \frac{1}{\sqrt[4]{\pi}}$ due to the normalization of the wavefunction. It is well-known that Gaussian wavepackets the coherent states of systems such as the free particle, the particle under a constant force, and the harmonic oscillator. Thus since we expect particles in coherent states to exhibit classical-like behavior, we expect particles with a Gaussian Wavepacket to have classical trajectories in Bohmian Mechanics.

5. A Free Particle With A Gaussian Wavepacket

Take an initial Gaussian Wavefunction

$$\psi(x,0) = \frac{A}{\sqrt{\sigma}} e^{\frac{-x^2}{2\sigma^2} + \frac{ip_0}{\hbar} x}$$

with a Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

In order to solve for the time-dependent wavefunction, we will first transform ψ into the momentum basis by the relation

$$\tilde{\psi}(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i(\frac{p}{\hbar})x} \psi(x,0)\, dx.$$

Solving this integral gives

$$\tilde{\psi}(p,0) = A\sqrt{\frac{\sigma}{\hbar}} e^{-(p-p_0)^2\sigma^2/2\hbar^2}$$

By solving the Schrodinger equation we get that

$$\tilde{\psi}(p,t) = \tilde{\psi}(p,0) e^{-(\frac{i}{\hbar})\frac{p^2}{2m}}$$

using the relation

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i(p/\hbar)x} \tilde{\psi}(p,t)\, dp$$

and defining

$$\beta = 1 + i\frac{\hbar t}{m\sigma^2}$$

we get that

$$\psi(x,t) = \frac{A}{\sqrt{\sigma\beta}} e^{f(x,t)}$$

where

$$f(x,t) = \frac{ip_0 x}{\hbar} - \frac{ip_0^2 t}{2m\hbar} - \frac{x^2}{2\sigma^2\beta} + \frac{xp_0 t}{2m\sigma^2\beta} + \frac{p_0^2 t^2}{4m^2\sigma^4\beta^2}$$

Knowing the time-dependent wavefunction, we can now solve for the Bohmian velocity field. We know that

$$\frac{\partial\psi(x,t)}{\partial x} = \frac{A}{\sqrt{\sigma\beta}} e^{f(x,t)} \frac{\partial f(x,t)}{\partial x}$$

Thus

$$\frac{\frac{\partial\psi(x,t)}{\partial x}}{\psi(x,t)} = \frac{\partial f(x,t)}{\partial x} = \frac{ip_0}{\hbar} - \frac{x}{\sigma^2\beta} + \frac{p_0 t}{2m\sigma^2\beta}$$

Therefore

$$\Im\{\frac{\partial\psi(x,t)}{\partial x}\} = \frac{p_0}{\hbar}$$

Thus we get from the guidance equation that

$$v_x(t) = \frac{p_0}{m}$$

and thus

$$x(t) = \frac{p_0 t}{m}.$$

Therefore the motions of a free particle with a Gaussian wavepacket is equivalent to the motion of a classical particle.

6. A Particle With A Constant Force And A Gaussian Wavepacket

Take an initial Gaussian wavefunction

$$\psi(x,0) = \frac{A}{\sqrt{\sigma}} e^{\frac{-x^2}{2\sigma^2} + \frac{ip_0}{\hbar}x}$$

under a constant force F. The Hamiltonian for such a system will be

$$\bar{H} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} - Fx$$

and thus by the Avron-Herbst formula [6] we know that

$$\psi(x,t) = \psi(x - \frac{F}{2m}t^2, 0)e^{-\frac{i}{\hbar}\frac{F^2}{6m}t^3 + \frac{i}{\hbar}tFx + \frac{i}{2m\hbar}t}$$

Therefore by substitution of our wavepacket into this equation we receive the following wavefunction

$$\psi(x,t) = \frac{A}{\sqrt{\sigma}} e^{\frac{-x^2}{2\sigma^2} + \frac{ip_0}{\hbar}x + \frac{-(\frac{F}{2m}t^2)^2}{2\sigma^2} + \frac{i\hbar}{p_0}\frac{F}{2m}t^2 - \frac{i}{\hbar}\frac{F^2}{6m}t^3 + \frac{i}{\hbar}tFx + \frac{i}{2m\hbar}t}$$

Thus

$$\frac{\partial\psi(x,t)}{\partial x} = (-\frac{x}{\sigma^2} + \frac{ip_0}{\hbar} + \frac{i}{\hbar}tF)\psi(x,t)$$

and

$$v_x(t) = \frac{\hbar}{m}\Im\{\frac{\frac{\partial\psi(x,t)}{\partial x}}{\psi(x,t)}\} = \frac{p_0}{m} + \frac{tF}{m}$$

Taking $F = ma_0$ we can write this equation as

$$v(t) = \frac{p_0}{m} + a_0 t$$

which is the velocity of the classical particle.

7. The Shifted Ground State Gaussian Wavefunction in the Harmonic Oscillator

Take an initial Gaussian Wavefunction

$$\psi(x,0) = \frac{A}{\sqrt{\sigma}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

with a Hamiltonian of

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

Take the case where

$$\sigma = \sqrt{\frac{\hbar}{m\omega}}$$

This wavepacket is equal to the shifted ground state of the harmonic oscillator. Thus the time-dependent wavefunction is found to be [5]

$$\psi(x,t) = \frac{A}{\sqrt{\sigma}} e^{f(x,t)}$$

where

$$f(x,t) = -\frac{1}{2}(\frac{x}{\sigma} - \frac{a}{\sigma}\cos(\omega t))^2 - i(\frac{1}{2}\omega t + \frac{xa}{\sigma^2}\sin\omega t - \frac{a^2}{4\sigma^2}\sin 2\omega t)$$

Notice that

$$\frac{\frac{\partial\psi}{\partial x}}{\psi} = \frac{\partial f(x,t)}{\partial x} = -\frac{x}{\sigma^2} - \frac{a}{\sigma^2}\cos(\omega t) - \frac{ia}{\sigma^2}\sin(\omega t)$$

and thus

$$v_x = \frac{\hbar}{m}\Im\{\frac{\frac{\partial\psi}{\partial x}}{\psi}\} = -\frac{\hbar a}{m\sigma^2}\sin(\omega t)$$

By plugging in σ we get that

$$v = a\omega\sin(\omega t).$$

which matches the classical result.

8. Conclusion

Bohmian Mechanics is a causal interpretation of Quantum Mechanics that was developed to answer philosophical issues dealing with the formulation of Quantum Mechanics. Coherent states of quantum systems have been analyzed before in Quantum Mechanics as being the states of least uncertainty and as state which exhibit classical-like behavior. By looking at these cases we notice that the coherent states of the free particle, the particle under a constant force, and harmonic oscillator all give trajectories that match the classical trajectory. This leads us to speculate that the trajectories of particles in coherent states may have classical trajectories.

References

[1] Bernstein, Jeremy. "More About Bohm's Quantum." American Journal of Physics 79.6 (2011): 601-06.
[2] Bowman, Gary E. "Bohmian Mechanics as a Heuristic Device: Wave Packets in the Harmonic Oscillator." American Journal of Physics 70.3 (2002): 313-18.
[3] Dürr, Detlef Christopher, and Stefan Christopher Teufel. Bohmian Mechanics: The Physics and Mathematics of Quantum Theory. Berlin: Springer, 2009.
[4] Holland, Peter R. The Quantum Theory of Motion: An Account of the De Broglie-Bohm Causal Interpretation of Quantum Mechanics. Cambridge [England: Cambridge UP, 1993.
[5] Schiff, Leonard I. Quantum Mechanics. New York: McGraw-Hill, 1968.
[6] Thaller, Bernd Christopher. Visual Quantum Mechanics: Selected Topics with Computer-Generated Animations of Quantum-Mechanical Phenomena. Page 193. New York: Springer/TELOS, 2000.